SOLUTIONS TO PRISM PROBLEMS

2015

1. (B)
Each number after 64 is one-half of the previous number.

2. (C)
$\frac{28}{3} = 9.3333$ is not a whole number. But 3 does divide all of the other given numbers.

3. (C)
If Andrew gives half of his cars (i.e. 2 cars) to Elizabeth and Liam gives one-half of his cars (i.e. 3 cars) to Elizabeth, they will then have 2+3= 5 cars between them while she will have $n + 2 + 3$ cars. We are given that $n + 2 + 3 = 2 \times 5$. Hence $n = 10 - 5 = 5$.

4. (A)
Let T be Tom's age and let M denote Mary's age.
We are given that $T = M + 2$ and $T + M = 20$. We now solve these two simulaneous equations. Substituting $M + 2$ for T in the second equation, we have $M + 2 + M = 20$. Hence $2M = 18$ so $M = 9$, and then $T = M + 2 = 9 + 2 = 11$. Finally the product of their ages is then $11 \times 9 = 99$.

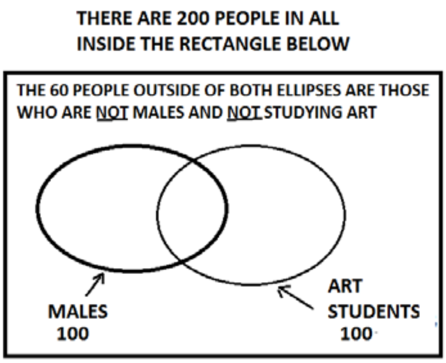
5. (A)
Since two chocolate bars currently cost €1, each bar costs €0.50 and if the price is increased by 10%, each bar will then cost $€0.50 + €0.05 = €0.55$. Similarly, each cake is currently $€2.10/3 = €0.70$ so if cake prices decrease by 10%, a cake will now cost $€0.70 - €0.07 = €0.63$. Hence Adam would now pay $3 \times €0.55 + 2 \times €0.63 = €1.65 + €1.26 = €2.91$ for 3 chocolate bars and 2 cakes.

6. (C)
Statement (1) is false because $2^2 + 2^3 = 4 + 8$ which does not equal $2^5 = 32$.
Statement (2) is true because $4(3(2-1)-2) = 4(3(1)-2) = 4(3-2) = 4(1) = 4$.
Statement (2) is false because $\frac{1}{3} + \frac{1}{3} = \frac{2}{3} \neq \frac{3}{2}$.
Finally, statement (4) is true because $21 > 20$ so $\frac{1}{21} < \frac{1}{20}$ and hence $\frac{1}{20} + \frac{1}{21} < \frac{1}{20} + \frac{1}{20} = \frac{2}{20} = \frac{1}{10}$.
We have thus shown that exactly two of the four statements are true.

7. (B)
One way of proceeding here is to first compare the three radicals $\sqrt{2}, \sqrt[3]{3}$ and $\sqrt[4]{4}$ by raising each to the power of 12 (this will ensure transformation of each of these numbers to integers); of course any postive number n is larger than another positive number m if and only if $n^{12} > m^{12}$.

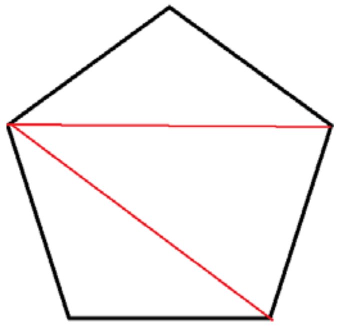
When raised to the power of 12, the three numbers $\sqrt{2}$, $\sqrt[3]{3}$ and $\sqrt[4]{4}$ are respectively $(\sqrt{2})^{12} = (2^{1/2})^{12} = 2^6 = 64$, $(\sqrt[3]{3})^{12} = (3^{1/3})^{12} = 3^4 = 81$ and $(\sqrt[4]{4})^{12} = (4^{1/4})^{12} = 4^3 = 64$.
The largest of these three numbers is 81, so $\sqrt[3]{3}$ is larger than each of $\sqrt{2}$ and $\sqrt[4]{4}$. We now claim that the second number on the list, $\frac{3}{2}$, is the largest of the five numbers. Since $\frac{3}{2} > 1$, it is sufficient for us to show that $\frac{3}{2} > \sqrt[3]{3}$. This inequality will be true if and only if (cubing both sides)
$(\frac{3}{2})^3 > (\sqrt[3]{3})^3$, i.e. if $\frac{27}{8} > 3$, and this is clearly true.

8. (C)
Many students will find it helpful to draw a Venn diagram like the following.

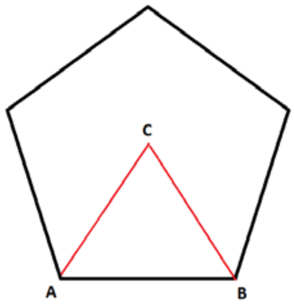


The heavier-shaded ellipse contains the 100 males, the lighter-shaded ellipse holds the 100 blue-eyed people, and all people outside of these two ellipses constitute all students in the class who are neither males nor study Art.
Since the total number of students who are neither males nor study Art is 60, there are $200 - 60 = 140$ students who are either male or study Art or are both male and study Art (that is, 120 students are in at least one of the two ellipses).
But there are 100 males and 100 Art students, so the overlap between males and Art students (that is the number of students in the intersection of the two ellipses, i.e. the number who were male Art students) must be
$100 + 100 - 140 = 60$.

9. (C)
One way of proceeding is to connect vertices of the decagon to form 3 triangles as shown below.
Clearly the sum of all 5 angles of the pentagon equals the sum of the angles in all 3 triangles. But the sum of the angles in the 3 triangles is $3 \times 180° = 540°$.
Thus the measure of the angle between any two adjacent sides of the pentagon is $\frac{540°}{5} = 108°$.



Another way:
Alternatively, the diagram below shows one of the 5 triangles that can be formed by joining the centre of the decagon to the 5 vertices.

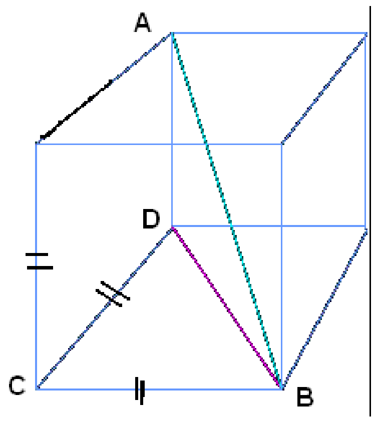


The measure of the angle $\angle ACB$ is $\frac{360°}{5} = 72°$ and since the three angles of the triangle sum to $180°$ and the triangle is isosceles, the measure of the angle $\angle BAC$ is $\frac{1}{2}(180° - 72°) = 54°$. Hence the measure of the angle between any two adjacent sides is $2 \times 54° = 108°$.

10. (E)
A little reflection shows that the 'worst case scenario' is where Jerome gets 10 blue socks in his first 10 selections. These will certainly be followed by 2 blue socks, so the maximum number of socks he must pack to ensure at least one pair of red and at least one pair of blue socks is 10+2 =12.

11. (C)
The simplest way to proceed is to express each of the three given numbers as some number to the power of 100. Already 7^{100} is of that form. Now $64^{50} = (8^2)^{50} = 8^{2\times 50} = 8^{100}$, and similarly, $3^{200} = 3^{2\times 100} = (3^2)^{100} = 9^{100}$. Thus the three given numbers $3^{200}, 64^{50}$ and 7^{100} are equal to $9^{100}, 8^{100}$ and 7^{100}, respectively. Clearly then $3^{200} > 64^{50} > 7^{100}$, which corresponds to choice (C).

12. (B)
The diagram below shows a cube with one of its diagonals AB and a diagonal DB of one of its faces.



Notice that by Pythagoras' Theorem applied to the triangle CDB, we have $\|DB\|^2 = \|CD\|^2 + \|CB\|^2$.

Thus $\|DB\|^2 = 2\|CD\|^2$ **(*)**

By Pythagoras' Theorem applied to the triangle ADB, we have

$$\|AB\|^2 = \|AD\|^2 + \|DB\|^2 \quad \textbf{(**)}$$

Inserting (*) into (**), we get $\|AB\|^2 = \|AD\|^2 + 2\|CD\|^2$ and hence

$$\|AB\|^2 = 3\|AD\|^2 \quad \textbf{(***)}$$

We now evaluate each of (A)-(E) and will express each as a square root for ease of comparison of their magnitudes.

The value of (A) is $2 \times 2 \times 2 = 8 = \sqrt{64}$.

For a cube described in (B), we have from (***) above that its volume is $\frac{4}{\sqrt{3}} \times \frac{4}{\sqrt{3}} \times \frac{4}{\sqrt{3}} = \frac{64}{3\sqrt{3}} = \sqrt{\frac{64 \times 64}{9 \times 3}} = \sqrt{\frac{4096}{27}}$.

Since the cube in (C) has surface area 20, the area of any one of its six faces is $\frac{20}{6}$ and hence the length of any one side is $\sqrt{\frac{20}{6}} = \sqrt{\frac{10}{3}}$.

Its volume is then $\left(\sqrt{\frac{10}{3}}\right)^3 = \sqrt{\left(\frac{10}{3}\right)^3} = \sqrt{\frac{1000}{27}}$.

The volume of the cube in (D) is $7 = \sqrt{49}$.

Finally, using (*) above, the length of each side of the cube in (E) is $\frac{3}{\sqrt{2}}$, and hence its volume is $\left(\frac{3}{\sqrt{2}}\right)^3 = \frac{27}{\sqrt{8}} = \sqrt{\frac{27 \times 27}{8}} = \sqrt{\frac{729}{8}}$.

We have shown that the magnitudes of (A), (B), (C), (D) and (E) are, respectively, $\sqrt{64}$, $\sqrt{\frac{4096}{27}}$, $\sqrt{\frac{1000}{27}}$, $\sqrt{49}$ and $\sqrt{\frac{729}{8}}$.

A comparison of these shows that the cube with the largest volume is the one described in (B). Note that the square roots can be dropped in comparing the magnitudes!

13. (D)

Let d be the distance from the bottom to the top of the hill so d is also the distance from the top to the bottom. Also let t_1 be the time taken to get from the bottom to the top of the hill, and let t_2 be the time taken to go from the top to the bottom. Let s be the speed at which the car was driven down the hill. Since speed is distance divided by time, we have $30 = d/t_1$ and $s = d/t_2$. Hence

$d = 30t_1 = st_2$. Accordingly, $t_2 = \frac{30}{s}t_1$.
Now the total distance is the distance to top + distance to the bottom = $30t_1 + st_2 = 2 \times 30t_1 = 60t_1$. The average speed, 40, is the total distance divided by the total time, i.e. twice the distance to the top divided by the total time. So we have $\frac{60t_1}{t_1 + t_2} = 40$. Substituting $t_2 = \frac{30}{s}t_1$, we find that $\frac{60t_1}{t_1 + \frac{30}{s}t_1} = 40$.
Dividing above and below by t_1, we have $\frac{60}{1 + \frac{30}{s}} = 40$. Hence $60 = 40(1 + \frac{30}{s})$, or $20 = \frac{1200}{s}$. Hence $s = \frac{1200}{20} = 60$.

Note: A study of the argument above shows that in computing average speed we do not use the arithmetic average of the speeds up and down the mountains, but rather the harmonic mean, $\frac{2}{\frac{1}{\text{speed up}} + \frac{1}{\text{speed down}}}$, which can be written as $\frac{1}{\frac{\frac{1}{\text{speed up}} + \frac{1}{\text{speed down}}}{2}}$, i.e. the reciprocal of the arithmetic mean of the reciprocals of the two speeds.That is,

$$\text{Average speed} = \frac{1}{\frac{\frac{1}{\text{speed up}} + \frac{1}{\text{speed down}}}{2}}$$

14. (E)
One way of solving this problem is to note that since $x - y = 10$, we have (on squaring both sides), $(x - y)^2 = 100$, i.e.
$x^2 + y^2 - 2xy = 100$. But $xy = 5$, so we have $x^2 + y^2 - 2(5) = 100$, and hence $x^2 + y^2 = 110$.

15. (C)
Let x be the speed of the boat in still water and let w be the speed of the water. The actual speed of the boat going down the river is then $x + w$ and the actual speed of the boat going up the river is $x - w$. Recall that average speed = $\frac{\text{distance travelled}}{\text{time taken}}$
so

distance travelled = (average speed)(time taken) **(*)**

Hence we have that

distance travelled down river = $2(x + w)$ **(**)**

and

distance travelled up river = $3(x - w)$ **(***)**

Since the distance travelled up the river is the same as the distance travelled down
the river, we see from (**) and (***) that $2(x + w) = 3(x - w)$. This is $2x + 2w = 3x - 3w$, so

$$x = 5w \qquad \textbf{(****)}$$

From (**) and (****), we have

distance travelled down river $= 2(5w + w) = 12w$ **(*****)**

[Of course, we could instead have used (***) and (****) to get to (*****).]

Then time taken for boat to travel from A to B in still water $\underset{\text{using (*)}}{=}$

$$\frac{\text{distance travelled to go from A to B in still water}}{\text{speed of boat on trip from A to B in still water}} \underset{\text{by (*****)}}{=} \frac{12w}{x} \underset{\text{by (****)}}{=} \frac{12w}{5w} = \frac{12}{5}$$

hours.

16. (D)

In the set $\{1, 2, \ldots, 1000\}$, let D_4 be the number of numbers that are divisible by 4, let D_6 be the number of numbers divisible by 6 and let $D_{4,6}$ be the number of numbers divisible by *both* 4 and 6. The number of number divisible by either 4 or 6 (that is, by at least one of the two numbers 4 and 6) is $D_4 + D_6 - D_{4,6}$. (This is because when we add the number of numbers divisible by 4 to the number of numbers divisible by 6, we have twice included the number of numbers divisible by both 4 and 6).

Now note that the number of numbers in a set $\{1, 2, \ldots, N\}$ that are divisible by an integer n is the smallest integer less than or equal to $\frac{N}{n}$; this number is commonly called the *floor of* $\frac{N}{n}$, symbolically denoted $\lfloor \frac{N}{n} \rfloor$.

Now D_4 is $\lfloor \frac{1000}{4} \rfloor = \lfloor 250 \rfloor = 250$ and D_6 is $\lfloor \frac{1000}{6} \rfloor = \lfloor 166.6667 \rfloor = 166$.

Next if a number is divisible by both 4 and 6 it must be divisible by 12 (the least common multiple of 4 and 6). Hence $D_{4,6} = \lfloor \frac{1000}{12} \rfloor = \lfloor \frac{1000}{4} \rfloor = \lfloor 83.3333 \rfloor = 83$.

Accordingly the desired answer is $D_4 + D_6 - D_{4,6} = 250 + 166 - 83 = 333$.

17. (E)

Trivially it is the case that 9 is a factor of 9.

Next note that $3^{100} - 27$ is a multiple of 9 because

$3^{100} - 27 = 3^{2(50)} - 9(3) = 9^{50} - 9 \times 3$ and each of the two numbers 9^{50} and 9×3 is a multiple of 9, so their difference is.

To see that 12345678987654321 is divisible by 9, one could carry out the division and find that the remainder is 0. However, it is a fact that an integer is divisible by 9 if and only if the sum of its digits is divisible by 9. The sum of the digits in 12345678987654321 is 81 which is indeed divisible by 9.

Analogously, 222222277777771 is **not** divisible by 9, either by carrying out the division or by noting that the sum of the digits in 222222277777771 is $2 \times 7 + 7 \times 7 + 1 = 64$, and 9 is not a factor of 64.

Since we have found that 9 does not divide 222222277777771, we need not check the remaining number $9^{8^{7^{6^{5^{4^{3^{2^{1}}}}}}}}$ (though it is obvious that 9 divides $9^{8^{7^{6^{5^{4^{3^{2^{1}}}}}}}}$ since this number is a power of 9.

18. (C)
We are given that the total mass of 100kg is partitioned as 1kg solids (1%) and 99kg water (99%). After evaporation, there will still be 1kg of solids and we are told that this is 2% of the entire content's mass (since we are told that 98% of the content is now water). So 2% of the content is 1kg in masst, and hence 100% of the content has mass $\frac{2\text{kg}}{0.02} = 50$kg.
Note: Many people find the answer to this question to be extremely surprising!

19. (E)
Since they arrived home 10 minutes earlier than usual, this means that Sean's wife chopped 10 minutes from her usual travel time to and from the station, or five minutes from her travel time to the station. It follows that she met Sean at 8:55pm, five minutes before his usual pick-up time . He started walking at 8pm; therefore he walked for 55 minutes.

20. (B)
The sister who was late could not have been Antonia, the truthful one, or else she would not have called herself Tosia. So either Dara or Tosia was late. But, if Tosia was late, then one of the other sisters, Antonia, would have truthfully answered that Tosia was late. She didn't, so it must be the case that Dara was late (and lied that her name was Tosia). Antonia then truthfully told that the name of the girl who was late is Dara, while Tosia lied that Antonia was late.
